University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

**Solution Set 7** (Compiled by Uday Varadarajan)

1. For an insulator with linear magnetic properties, the bound current is given by

$$J_b = \nabla \times \vec{M} = \nabla \times \chi_m \vec{H} = \chi_m J_f = 0, \tag{1}$$

where $J_f = 0$ since the material is an insulator, which does not support free, time independent flowing currents.

2. and

3. (a) The normal to the sphere is $\hat{\mathbf{r}}$, so the bound surface current is given by,

$$\vec{K}_b = \vec{M} \times \hat{\mathbf{n}} = M\hat{\mathbf{z}} \times \hat{\mathbf{r}} = M sin\theta\hat{\boldsymbol{\phi}}. \tag{2}$$

Since $\nabla \cdot \vec{M} = 0$ inside the sphere, there is no volume current density.

(b) The magnetic field of an ideal magnetic dipole is given by Griffiths 5.87,

$$\vec{B}(\vec{r}) = \frac{\mu}{4\pi r^3}(3(\vec{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{m}) = \frac{\mu M_0 b^3}{3r^3}(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}). \tag{3}$$

(c) The boundary conditions state,

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{K}_b \times \hat{\mathbf{n}} \tag{4}$$

so we have,

$$\vec{B}_{below} = \frac{\mu_0 M_0}{3}(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}) - \mu_0 M sin\theta\hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \mu_0 M_0(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}} - \frac{1}{3}\hat{\mathbf{z}}) = \frac{2\mu_0 M_0}{3}\hat{\mathbf{z}}. \tag{5}$$

(d) Inside the sphere, we have,

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} = \frac{2M_0}{3}\hat{\mathbf{z}} - M_0\hat{\mathbf{z}} = -\frac{M_0}{3}\hat{\mathbf{z}}. \tag{6}$$

Now, outside the sphere, $\vec{M} = 0$, so $\vec{H} = \vec{B}/\mu_0$ and we can use the expression we derived in part (a), evaluated in the regions above ($\theta = 0, r = |z|$, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$) and below ($\theta = \pi$, $r = |z|$, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$) the sphere,

$$\begin{aligned}\int_{-\infty}^{\infty} \vec{H} \cdot \hat{\mathbf{z}} dz &= -\int_{-b}^{b} \frac{M_0}{3} dz + \int_{-\infty}^{-b} \frac{\mu M_0 b^3}{3|z|^3}(3(\cos\pi)(-\hat{\mathbf{z}}) - \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} dz + \int_{b}^{\infty} \frac{\mu M_0 b^3}{3|z|^3}(3\hat{\mathbf{z}} - \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} dz \\ &= -\frac{2bM_0}{3} + 2\int_{b}^{\infty} \frac{2\mu M_0 b^3}{3|z|^3} dz = -\frac{2bM_0}{3} + \frac{2M_0 b}{3} = 0.\end{aligned} \tag{7}$$

(e) We use the divergence theorem for an infinitessimal Gaussian pillbox P with boundary B with top and bottom surface area $A$, and the expressions for $\vec{H}$ above to find,

$$\int_P \nabla \cdot \vec{H} = \int_B \vec{H} \cdot d\vec{a} = A(\vec{H}_{above} - \vec{H}_{below}) \cdot \hat{\mathbf{z}} = A(\frac{2M_0}{3} + \frac{M_0}{3}) = AM_0. \tag{8}$$

Thus, we see that $\nabla \cdot \vec{H}$ cannot possibly vanish everywhere, as its integral over this pillbox does not vanish.

4. (a) First, we compute the bound surface current around the rim of the nickel, expressed in terms of cylindrical coordinates,

$$\vec{K}_b(s = a, |z| < d) = \vec{M} \times \hat{\mathbf{n}} = M_0\hat{\mathbf{z}} \times \hat{\mathbf{s}} = M_0\hat{\boldsymbol{\phi}}. \tag{9}$$

Thus, we can think of the magnetized nickel as a bunch of current loops circulating around its edge. To find the result for the magnetic field at the origin, we can integrate the result (Griffiths 5.38) for the magnetic field due along the $z$-axis for a single current loop from $z = -d/2$ (this represents the contribution of the uppermost loop to the magnetic field at the origin) to $z = d/2$ (the contribution of the lowest loop), where $dI = M_0 dz$,

$$B(0,0,0) = \int dB = \int \frac{\mu_0 dI}{2} \frac{a^2}{(a^2+z^2)^{3/2}} = \frac{\mu_0 M_0 a^2}{2} \int_{-d/2}^{d/2} \frac{dz}{(a^2+z^2)^{3/2}} = \frac{\mu_0 d M_0}{2\sqrt{a^2+d^2/4}} \approx \frac{\mu_0 M_0 d}{2a} - \frac{\mu_0 M_0 d^3}{16a^3}. \tag{10}$$

Clearly, only the first term is needed, but we write out the second term for future reference. As the magnetic field always points in the $\hat{\mathbf{z}}$ direction along the $z$-axis for a current loop, it also points upwards here.

(b) The magnetic flux through a loop of radius $s$ can be determined by doing an integral of $\vec{A}$ over the loop. As $\vec{A}$ is evaluated only at $s = b \gg a$ on the loop, the first order contribution in $d/b \ll a/b \ll 1$ to $\vec{A}$ comes from the dipole term. The dipole moment of the nickel is $\vec{m} = \pi a^2 dM\hat{\mathbf{z}}$, giving rise to the vector potential of Griffiths 5.83,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathbf{r}}}{r^2}. \tag{11}$$

We can use this expression to compute the magnetic flux,

$$\Phi_B = \oint_{s=b} A \cdot d\vec{l} = \frac{\mu_0}{4\pi} \int_{0,s=b}^{2\pi} \frac{\pi a^2 dM}{s^2} (\hat{\mathbf{z}} \times \hat{\mathbf{s}}) \cdot \hat{\boldsymbol{\phi}} d\phi = \frac{\mu_0}{2} \frac{\pi a^2 dM}{b^2}. \tag{12}$$

5. (a) Using cyclindrical symmetry, we know that $\vec{H}$ and the magnetic field is always in the $\hat{\boldsymbol{\phi}}$ direction and independent of $z$. Thus, we apply Ampere's Law to circles centered at the origin in the $z = 0$ plane of successively larger radii,

$$\oint_{<a} \vec{H}_{<a} \cdot d\vec{l} = 2\pi s H_{<a}(s) = I_{f,enc} = I\left(\frac{s^2}{a^2}\right) \Rightarrow H_{<a}(s) = \frac{Is}{2\pi a^2}, \tag{13}$$

$$\oint_{>a} \vec{H}_{>a} \cdot d\vec{l} = 2\pi s H_{>a}(s) = I_{f,enc} = I \Rightarrow H_{>a}(s) = \frac{I}{2\pi s}, \tag{14}$$

(b) Now, neglecting the weak diamagnetism of copper, we expect that for the regions $s < a$ and $s > b$, $\vec{M} = 0$ and $\vec{B} = \mu_0 \vec{H}$, while for $a < s < b$ we have $\vec{B} = \mu \vec{H}$ and $\vec{M} = (\mu_r - 1)\vec{H}$. Thus, we also get that $\vec{B}$ and $\vec{M}$ point in the $\hat{\boldsymbol{\phi}}$ direction and have magnitudes given by,

$$B_{<a}(s) = \frac{\mu_0 Is}{2\pi a^2} = M_{<a}(s) = 0, \tag{15}$$

$$B_{a<s<b}(s) = \frac{\mu I}{2\pi s} = M_{a<s<b}(s) = \frac{(\mu_r - 1)I}{2\pi s}, \tag{16}$$

$$B_{>d}(s) = \frac{\mu_0 I}{2\pi s} = M_{>b}(s) = 0. \tag{17}$$

(c) The only region in which we have non-vanishing $\vec{M}$ is the region $a < s < b$, so the bulk bound current density there is,

$$\vec{J}(s) = \nabla \times \vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\hat{\mathbf{z}} = 0. \tag{18}$$

We get surface bound current densities from the surface $s = a$ (where we need to take the inward pointing normal vector) and $s = b$ (the outward pointing normal vector) given by,

$$\vec{K}_{s=a} = \vec{M}(s=a) \times (-\hat{\mathbf{s}}) = -\frac{(\mu_r - 1)I}{2\pi a} \hat{\boldsymbol{\phi}} \times \hat{\mathbf{s}} = \frac{(\mu_r - 1)I}{2\pi a}\hat{\mathbf{z}} \tag{19}$$

$$\vec{K}_{s=b} = \vec{M}(s=b) \times \hat{\mathbf{s}} = \frac{(\mu_r - 1)I}{2\pi b} \hat{\boldsymbol{\phi}} \times \hat{\mathbf{s}} = -\frac{(\mu_r - 1)I}{2\pi b}\hat{\mathbf{z}}. \tag{20}$$

(d) Outside the Ferrite,

$$\mu_0 \vec{B} = \vec{H} \Rightarrow \oint \mu_0 \vec{B} \cdot d\vec{l} = I_f + I_b = \oint \vec{H} \cdot d\vec{l} = I_f \Rightarrow I_b = 0 \tag{21}$$

We can verify this using the bound surface densities computed above,

$$I_b = 2\pi a \vec{K}_{s=a} \cdot \hat{\mathbf{z}} + 2\pi b \vec{K}_{s=b} \cdot \hat{\mathbf{z}} = (\mu_r - 1)I - (\mu_r - 1)I = 0. \tag{22}$$

6. We use the hysteresis curve of Fig 6.29, and note that the top curve describes our system - it has been run into saturation on the far right, and is brought back down to zero $H \propto I$. The residual magnetic field inside the cylindrical rod can be read off by approximating the $y$-intercept of curve, which is about $B = 0.75T$. However, there is a gap in this configuration, and we want the magnetic field in that gap. If the gap was not present, this would be just $B = 0.75$. So, we model this gap by a superposition of a full toroidal iron rod and a nickel-like chunk of material with a constand but opposite magnetization $M = -B/\mu_0 = -0.75T/\mu_0$. Now, we just need to subtract the contribution of this nickel to the magnetic field at the origin - but we calculated this to the required accuracy in problem 4a, with $d = 0.1cm$ and $a = 1cm$, and with ten percent accuracy,

$$\Delta B = \frac{\mu_0 M_0 d}{2a} = -0.75T \times (0.1cm)/(2cm) = -0.038T \Rightarrow B = 0.75T - 0.038T \approx 0.7T. \tag{23}$$

7. Just as above, we think of this system as a toroid with constant magnetization superposed with a nickel-like chunk with the opposite magnetization. The toroid has a constant magnetization $\vec{M} = M_0 \hat{\boldsymbol{\phi}}$. Since the normal vector to the toroid at its surface is always perpendicular to $\vec{M}$, we get a constant surface current density of magnitude $K = |\vec{M} \times \hat{\mathbf{n}}| = M_0$ circulating around the axis of the rod[1]. Thus, we are exactly in the situation of Griffiths example 5.10, and using the fact that the total current enclosed by the Amperian loop is $2\pi s M_0$, we have

$$B = \frac{\mu_0 (2\pi s M_0)}{2\pi s} = \mu_0 M_0. \tag{24}$$

Now, we just need to remove the effect of the nickel-like chunk, which we can do just as above, but including an extra term,

$$\Delta B = \frac{\mu_0 M_0 d}{2a} - \frac{\mu_0 M_0 d^3}{16a^3} = -\mu_0 M_0 \times (\frac{1}{20} - \frac{1}{16000}) \Rightarrow B = \mu_0 M_0 \times (1 - \frac{1}{20} + \frac{1}{16000}) \approx 0.95 \mu_0 M_0. \tag{25}$$

8. We can undo the magnetization in at least a couple of different ways. One way to do this is to apply negative and positive currents in alternating patterns with amplitudes well below those needed for saturation. As we expect the curves on the way up to always stay below the curves on the way down, this will eventually lead to lower and lower magnetizations at zero current, until we finally hit zero magnetization at zero current. Another way to undo the magnetization is to heat up the iron above its Curie temperature.

[1] In making this statement, we have really made an approximation - bending the rod into a C should have resulted in a profile in which the magnetizations are slightly lower on the outside, thereby resulting in surface currents densities that vary accordingly.